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SHORTER COMMUNICATION

HEATED TWO-DIMENSIONAL JET DISCHARGED AT WATER SURFACE

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NOMENCLATURE

b_o, nozzle width;
Re, Reynolds number, $\equiv u_o b_o/\nu$;
Ri, local Richardson number, $\equiv (g\partial\rho/\partial y)/(\rho(\partial u/\partial y)^2)$;
Ri_o, nozzle-exit Richardson number, $\equiv (\rho_w - \rho_o)gb_o/(\rho u_o^2)$;
t, temperature;
T, dimensionless temperature, $\equiv (t-t_w)/(t_o-t_w)$;
u, v, velocity components;
U, V, dimensionless velocity components, $\equiv u/u_o,\ vb_o/\varepsilon_{mn}$;
x, y, coordinates;
X, Y, dimensionless coordinates,
$$\equiv \int_0^x \varepsilon_{mn}/(u_o b_o^2)\,dx,\ y/b_o;$$
X_b, dimensionless coordinate, $\equiv x/b_o$;
$Y_{U1/2}$, $Y_{T1/2}$, half widths for the velocity and temperature distributions, respectively;
ε_m, ε_t, eddy viscosity and conductivity;
ε_{mn}, ε_{tn}, eddy viscosity and conductivity at neutral stratification;
Ψ_∞, stream function at the jet boundary edge;
ω, stream function normalized by Ψ_∞.

INTRODUCTION

INVESTIGATIONS on buoyant jets discharged at the water surface are relevant to thermal pollution and sewage disposal, in which the density stratification exercises essential effects on the flow and temperature fields through the change of turbulent transports of momentum and heat. The heated two-dimensional jet discharged at the water surface, which is schematically illustrated in Fig. 2, has been studied by Wada *et al.* Wada [1] made a theoretical analysis, and calculated the velocity and temperature distributions for the case corresponding to his field survey. He employed the Mamayev's empirical formulas [14] for the eddy viscosity and conductivity at stable stratification, which were derived on the basis of Jacobsen's data [15] obtained at very high Richardson numbers, $Ri = 2{\cdot}6$–30, and give several times larger values of the eddy viscosity than the data measured by other investigators for Ri less than 2. He presented the solution of the third approximation obtained by an iterative method, which agrees with the result of his field survey qualitatively but not quantitatively. He [2] also measured some representative velocity, which is neither the mean nor surface velocity, and the jet width, which was determined photographically by adding a dye to the discharged water. The jet width measured by the dye method is indefinite and somewhat larger than the half width according to the experiment by Stefan [5]. Tamai [4] solved the momentum equation by the integral method with the entrainment coefficient which includes the effect of stratification. His mean velocity and boundary-layer thickness were compared with the aforementioned Wada's measurements [2]. There is a good agreement between them, which is erroneous because of the indefiniteness of Wada's velocity and jet width. Tamai [3] measured the distributions of velocity and density of pure water discharged into salt water. Stefan [5] measured the onset of interfacial instability, assymptotic total flow rate, boundary-layer thickness by the dye method, and the local velocity and temperature distributions at $X_b = 353$ for $Ri_o = 0{\cdot}0595$, $Re = 196$ and $t_o = t_w = 10°F$.

The present paper investigates theoretically the flow and thermal characteristics, varying Ri_o. The water density varies almost linearly with temperature when the temperature level is sufficiently higher than that of the point of maximum density, and the temperature difference is small. Let us call this situation "the summer condition". When the heated water is discharged into receiving water of a temperature lower than the point of maximum density, however, the density variation is no longer linear, and the behavior of jet changes significantly. This may occur in winter, and is termed "the winter condition". An example solution is also presented for this case.

EDDY DIFFUSIVITIES AT STRATIFICATION

When a flow is stratified, the gravitation plays a dominant role in the turbulent diffusion. The stratification is classified into the stable stratification where the density gradient in the direction of gravity is positive, neutral stratification where there is no density gradient, and unstable stratification where the density gradient is negative. The effects of stratification on diffusion are different for stable and unstable stratification. At stable stratification, fluid particles must do work against the gravitational or buoyancy force when they are transported in the vertical direction. Therefore, the turbulent mixing process is significantly impeded, and the eddy diffusivities are greatly reduced. On the other hand, the unstable stratification enhances the turbulent diffusion. The stability or instability of stratification is characterized by the Richardson number which governs turbulent diffusion.

Density stratification occurs in oceanographic and meterological phenomena, and the eddy diffusivities at stratification have been extensively investigated in conjunction with them. However, the data scatter so much owing to the complexity of phenomena that it is impossible to draw a curve which fits all the data reasonably well. The author, therefore, attempted to derive empirical formulas which agree with the mean values of all the available data of Pasquill [6], Rider [7], Swinbank [8], Ellison and Turner [9], Sjöberg [10], and Oke [11] for stable stratification, i.e. $Ri > 0$. The mean values were taken as the arithmetic average of the data which fall within ranges of 0·1 of Ri, and the result is shown in Fig. 1. The ratios of eddy viscosity to that at neutral stratification, i.e. $Ri = 0$, and eddy conductivity to eddy viscosity are well correlated by

$$\frac{\varepsilon_m}{\varepsilon_{mn}} = \frac{\varepsilon_t}{\varepsilon_m} = \frac{1}{1+10\,Ri}. \tag{1}$$

There are additional data available for values of Ri in the range of 2–30, and the above formula lies in between them. However, $\varepsilon_m/\varepsilon_{mn}$ and $\varepsilon_t/\varepsilon_m$ are of the order of less than 0·01 in this range of Ri so that molecular diffusion is dominant, and the error of the formulas is insignificant. It is controversial what value $\varepsilon_t/\varepsilon_m$ takes on at neutral stratification. Ellison and Turner compared the results of measurements by several investigators, and found that it varied from 0·8 to 1·5. They concluded that it lay between 1·3 and 1·4, which agrees with their own experiment. However, the majority of those who investigated the eddy diffusivities at stratification verified by their data that $\varepsilon_{tn}/\varepsilon_{mn} = 1$. This value will be adopted, resulting in the equation

$$\frac{\varepsilon_t}{\varepsilon_{tn}} = \frac{1}{(1+10\,Ri)^2}. \tag{2}$$

For unstable stratification, i.e. $Ri < 0$, we use Gurvich's formula [12]

$$\frac{\varepsilon_m}{\varepsilon_{mn}} = \frac{\varepsilon_t}{\varepsilon_{tn}} = \begin{cases} 1 & \text{for } |Ri| \leqslant 0{\cdot}0386, \\ 5{\cdot}09\sqrt{(|Ri|)} & \text{for } |Ri| > 0{\cdot}0386, \end{cases} \tag{3}$$

which is a good representation of the mean values of all the past data including his own.

ANALYSIS

The turbulent boundary-layer equations are given as follows

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left\{(\nu+\varepsilon_m)\frac{\partial u}{\partial y}\right\}, \tag{5}$$

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \frac{\partial}{\partial y}\left\{(\alpha+\varepsilon_t)\frac{\partial t}{\partial y}\right\}, \tag{6}$$

where it is assumed that the pressure is a function of y alone, and is balanced by the gravitational force. A sketch of the jet discharge is shown in Fig. 2. Equations (4)–(6) are non-dimensionalized by the use of the dimensionless variables defined in the nomenclature.

The non-dimensional equations are solved by Spalding–Patankar's method [13], which transforms the equations to the following form

$$\frac{\partial \Phi}{\partial X} - b\omega\frac{\partial \Phi}{\partial \omega} = \frac{\partial}{\partial \omega}\left(c\frac{\partial \Phi}{\partial \omega}\right). \tag{7}$$

Equation (7) is subject to the boundary conditions

$$\left.\begin{aligned} &\Phi = 1 \quad \text{for} \quad 0 \leqslant \omega \leqslant 1 \quad \text{at} \quad X = 0, \\ &\frac{\partial \Phi}{\partial \omega} = 0 \quad \text{for} \quad X \geqslant 0 \quad \text{at} \quad \omega = 0, \\ &\Phi \to 0 \quad \text{for} \quad X \geqslant 0 \quad \text{as} \quad \omega \to 1. \end{aligned}\right\} \tag{8}$$

Φ is a dummy variable which represents either U or T, and

$$b \equiv \frac{1}{\Psi_\infty}\frac{d\Psi_\infty}{dX}, \quad c \equiv \begin{cases} \dfrac{1}{\Psi_\infty^2}\left(\dfrac{\nu}{\varepsilon_{mn}} + \dfrac{\varepsilon_m}{\varepsilon_{mn}}\right)U & \text{for } \Phi = U, \\ \dfrac{1}{\Psi_\infty^2}\left(\dfrac{\nu}{Pr\,\varepsilon_{mn}} + \dfrac{\varepsilon_t}{\varepsilon_{tn}}\right)U & \text{for } \Phi = T. \end{cases} \tag{9}$$

The water density is approximated by a polynomial of 9th order, using the density–temperature data [16]. The error of approximation is less than 0·1 per cent. Prandtl's second hypothesis is employed for the eddy viscosity at neutral stratification. By the use of Reichardt's experiment, we get

$$\frac{\varepsilon_{mn}}{\nu} = 6{\cdot}78 \times 10^{-2}\,Re\sqrt[3]{(X)}. \tag{10}$$

$\partial\Phi/\partial X$ is approximated by the conventional forward difference. Since c tends to be small at stable stratification, the forward difference is also utilized for $\partial\Phi/\partial\omega$ in order to insure the stability of the difference equations. The central difference is applied to $\partial(c\,\partial\Phi/\partial\omega)/\partial\omega$, and c is evaluated one step upstream for linearization. Non-uniform mesh sizes are used in both the X- and ω-directions. The mesh size in the X-direction used here is one order smaller than that of Spalding–Patankar, while the interval in the ω-direction is divided into 40 meshes. The iteration to determine Ψ_∞ is terminated when the difference of Ψ_∞ between the $(m+1)$-th and m-th iteration divided by Ψ_∞ at the $(m+1)$-th iteration becomes less than 10^{-6}.

RESULTS AND DISCUSSION

The present phenomenon is governed by four dimensionless parameters, Re, Pr, Ri_o, and the density variation with temperature. The effects of the first two are small in the turbulent water jet treated here ($Re \gtrsim 10\,000$ and $Pr > 1$), and, when t_w and t_o are specified, the only controlling parameter is Ri_o. The author chose two temperature ranges, i.e. $t_w = 25°C$, $t_o = 35°C$ and $t_w = 1°C$, $t_o = 11°C$. The density varies almost linearly with t in the former case. In the latter case, however, the density does not vary monotonously with t, and has a maximum point at 4°C. Therefore, the behavior of the jet is essentially different from the former case even if Ri_o is the same. The range of Ri_o is 0·06–0·25 for power plants in operation according to Wada [2]. It is varied from 0 to 0·6 in the present analysis, while Re and Pr are fixed at 10 000 and 5·69 at 30°C respectively. Ri_o for the latter temperature-range is about 0·032 if the average nozzle width and nozzle-exit velocity for the summer condition are used, and $Pr = 11{\cdot}0$ at 6°C, which corresponds to the situation in winter.

As mentioned in the introduction, Wada's measurements [2] are indefinite in concepts. In Wada's, Tamai's [3] and Stefan's [5] experiments the boundary-layer thickness is an independent parameter controlled by the downstream tailgate, while it is a dependent variable in the flow configuration of the present analysis. It is impossible, therefore, to compare the present theoretical results with the past experiments.

Figure 2 shows the variation of surface velocity and temperature with X_b. U_s and T_s for $Ri_o = 0$ varies as $1/\sqrt{(X_b)}$ in the fully developed region. They decrease more slowly as Ri_o increases. The effect of stratification on T_s is more pronounced, and there is almost no recognizable decrease of T_s for $Ri_o \geqslant 0{\cdot}07$.

The total flow rate or stream function at the boundary edge is also illustrated in Fig. 2. Ψ_∞ increases with X_b

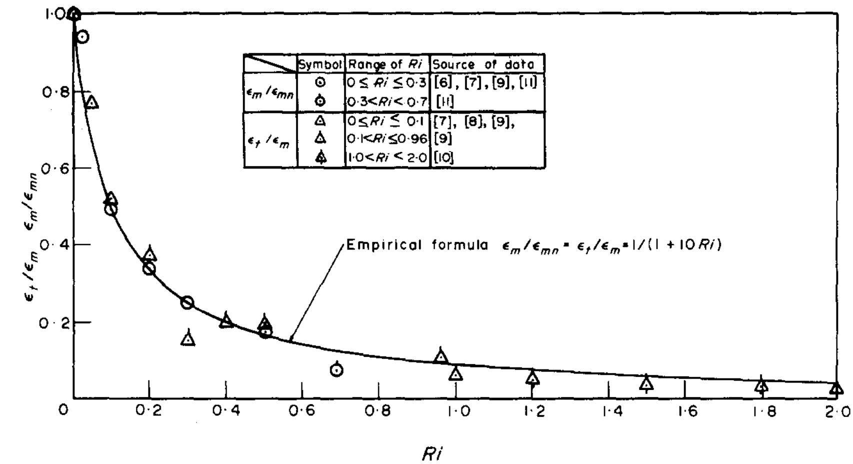

FIG. 1. Eddy diffusivities at stable stratification.

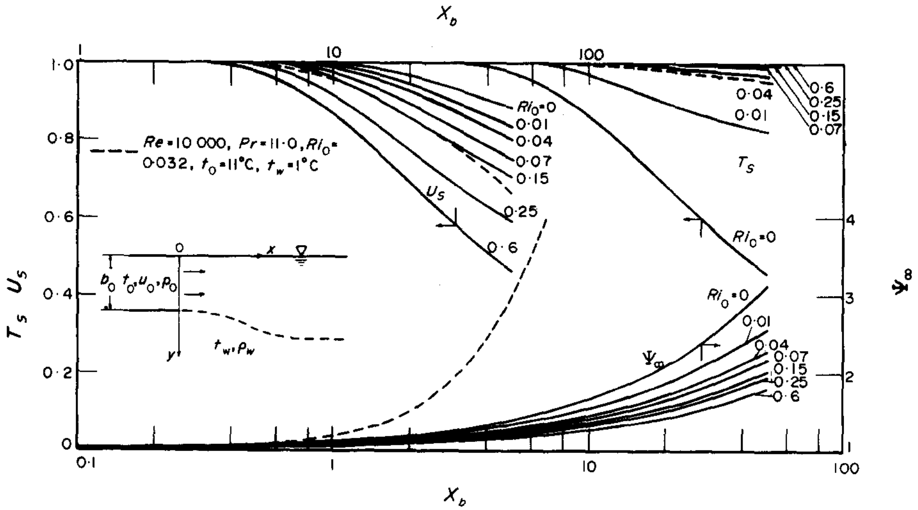

FIG. 2. Surface velocity, surface temperature and flow rate ($Re = 10\,000$, $Pr = 5{\cdot}69$, $t_o = 35°C$, $t_w = 25°C$).

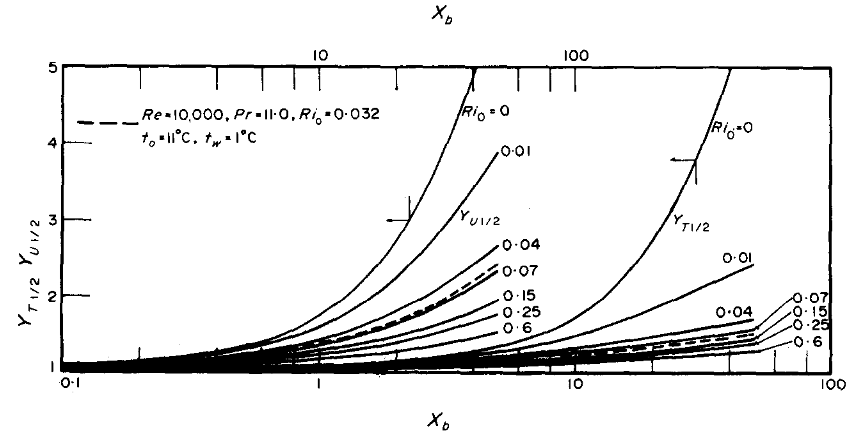

FIG. 3. Half widths for velocity and temperature distributions ($Re = 10\,000$, $Pr = 5{\cdot}69$, $t_o = 35°C$, $t_w = 25°C$).

because the receiving water is entrained by the jet through shearing stress, and it varies in proportion to $\sqrt{(X_b)}$ for $Ri_o = 0$. However, the entrainment is reduced by stable stratification so that Ψ_∞ decreases with Ri_o.

Figure 3 shows the half widths for the velocity and temperature distributions. For $Ri_o = 0$, they increase linearly with X_b. Since the mixing of the jet with the receiving water is reduced at stable stratification, $Y_{U1/2}$ and $Y_{T1/2}$ decrease with Ri_o. The decrease of $Y_{T1/2}$ is more remarkable, and $Y_{T1/2}$ tends to be almost constant with X_b for large Ri_o.

The behavior of the jet for the winter condition is illustrated by broken lines in Figs. 2 and 3. Apparently, U_s and T_s are not affected in the vicinity of the nozzle exit by the unstable stratification in the boundary-edge region, and lie between those for $Ri_o = 0{\cdot}01$ and $0{\cdot}04$. Far downstream the boundary-edge region penetrates into the surface region, and U_s exhibits a trend to fall rapidly. The penetration is not remarkable for t, and T_s remains almost unchanged. Since entrainment takes place at the boundary edge, and is directly affected by unstable stratification, Ψ_∞ increases significantly. $Y_{U1/2}$ and $Y_{T1/2}$ also show marked effects of unstable stratification. Their curves lie between those for $Ri_o = 0{\cdot}01$ and $0{\cdot}04$ close to the nozzle exit, but they decrease downstream much more than U_s and T_s because the half widths are located closer than the surface to the boundary edge. The velocity and temperature decrease with y almost at the same rate in the boundary-edge region owing to the equality of ε_m and ε_t at unstable stratification. However, the temperature decreases less than the velocity in the surface region because ε_t is much smaller than ε_m at stable stratification. Therefore, $Y_{T1/2}$ is much smaller and lies between those for $Ri_o = 0{\cdot}07$ and $0{\cdot}15$ for $X_b \gg 1$, while $Y_{U1/2}$ lies between those for $Ri_o = 0{\cdot}04$ and $0{\cdot}07$.

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